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Supersymmetric one-parameter strict isospectrality for the attractive δ potentials

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Abstract

The Schrödinger equation with attractive δ potential has been previously studied in the supersymmetric quantum mechanical approach by a number of authors, but they all used only the particular superpotential solution. Here, we introduce a one-parameter family of strictly isospectral attractive δ function potentials, which is based on the general superpotential (general Riccati) solution, we study the problem in some detail and suggest possible applications.

The $\delta(x)$ (pseudo)potential is a well known 'zero-range' potential with applications in solid state physics [1] and many other areas. It has been used as a textbook example for many mathematical procedures in quantum mechanics. One such technique, Witten's supersymmetric scheme [2], has been employed for the attractive delta potential by several authors [3, 4, 5]. However, in all those studies there is a missing point, namely all the authors so far used only the particular Witten superpotential W_0, which is related to the ground state wavefunction in the well-known way $u_0 = e^{-\int^x W_0}$, and no mention is made of the general superpotential, i.e., the general Riccati solution for the δ potential case. In this work we present the supersymmetric approach to the attractive delta potential problem based on the general superpotential.

To help the reader to better understand our problem we start with its underlying mathematical scheme. Thus, we consider a Riccati equation (RE) of the type $W^{'} = -W^2 + V_2(x)$ for which we suppose to know a particular solution W_0. Let $W_1 = W_0 + u$ be the second solution. By substituting W_1 in RE one gets the Bernoulli equation $u^{'} = -u^2 - 2W_0 u$, which by means of $u = 1/v$ is turned into the first order linear differential equation $v^{'} - 2W_0 v - 1 = 0$. The latter one can be solved by employing the integration

factor $f_0 = e^{-2\int^x W_0}$, leading to the solution $v = f_0^{-1}(C + \int^x f_0)$, where C is an arbitrary integration constant. Coming back to the general Riccati solution, one gets

$$W_1 = W_0 + \frac{f_0}{C + \int^x f_0} = W_0 + \frac{d}{dx}\left[\ln(C + \int^x f_0)\right] . \tag{1}$$

The point now is that in the process of factorizing the one-dimensional Schrödinger operator $-d^2/dx^2 + V_1(x)$ the aforementioned Riccati solutions occur in the non-operatorial part of the factorization operators as follows. W_0 occurs in the case of Witten's factorization [2] $(-d/dx + W_0)(d/dx + W_0)(\equiv A_0^\dagger A_0)$, whereas W_1 occurs for Mielnik's factorization [6] $(-d/dx + W_1)(d/dx + W_1)(\equiv A_1^\dagger A_1)$. Notice that $[A_0^\dagger, A_0] = 2W_0'$, whereas $[A_1^\dagger, A_1] = 2W_1^{'}$. We further notice that $\sqrt{f_0}$ is the ground state (nodeless) wavefunction of V_1 and $\Delta V_0 = -2W_0^{'}$ is the Darboux transform contribution to the potential V_1, leading to a new potential $V_{1,D0} = V_1 - 2W_0^{'} \equiv V_2$, which in supersymmetric quantum mechanics is known as the supersymmetric partner of the initial potential V_1. Even more interesting is that $\frac{\sqrt{f_0}}{C+\int^x f_0}$ can be interpreted as the ground state wavefunction corresponding to Mielnik's superpotential (see below), and $\Delta V_1 = -2W_1^{'}$ can be thought of as the general Darboux transform part in the potential. Therefore, there is a one-parameter family of Darboux potentials given by $V_{1,D1} = V_1 - 2W_1^{'}$, which are strictly isospectral to the initial one, in the sense that each member of the family has the same supersymmetric partner V_2 and the same energy eigenvalues and scattering amplitudes as V_1. In terms of the ground state wavefunction of V_1, $\psi_0 = \sqrt{f_0}$, each member of the strictly isospectral family of potentials reads

$$V_{iso;i} = V_1 + \Delta V_1 = V_1(x) - 2\frac{d^2}{dx^2}\ln\left(C_i + \int^x f_0\right) \tag{2}$$

or

$$V_{iso;i} = V_1(x) - \frac{4\psi_0\psi_0^{'}}{C_i + \int^x \psi_0^2} + \frac{2\psi_0^4}{(C_i + \int^x \psi_0^2)^2} . \tag{3}$$

For all half-line potentials the lower limit of the integral term is zero, whereas for the full-line potentials is $-\infty$. The ground state wavefunctions of this family are of the type $\psi_{0,iso} = \frac{\psi_0}{(C+\int^x \psi_0^2)}$. Indeed, one can write

$$W_1 = -\frac{d}{dx}\ln\left[\frac{\psi_0}{(C + \int^x \psi_0^2)}\right] = -\frac{d}{dx}\ln\psi_{0,iso} , \tag{4}$$

which is the supersymmetric formula introducing the superpotential in terms of the ground state wavefunction. If one consider these isospectral functions as quantum mechanical wavefunctions, the problem of the normalization constant should be contemplated. It is easy to see that the normalization constant is $N_{iso} = \sqrt{C(C+1)}$ [7] and as such C is not allowed to be in $[-1,0]$. The $C=0$ limit is known as the Pursey limit [8], whereas the $C=-1$ limit is the Abraham-Moses limit [9]. However, in the present work we shall consider both the case with the normalization constant included and the case without it.

Let us now pass to the attractive $g\delta(x)$ potential, where $g<0$ gives the strength of the interaction (the binding power). It has been shown that $W_0 = \frac{g}{2}\mathrm{sign(x)}$ [4]. In other words, $A_0 = d/dx + \frac{g}{2}\mathrm{sign(x)}$ and $A_0^\dagger = -d/dx + \frac{g}{2}\mathrm{sign(x)}$. Indeed, one cannot use the Heaviside step function as the superpotential since its square is not a constant. Therefore, one should work with the sign function, which is a combination of step functions. $A^\dagger\psi_0 = 0$ implies $\psi_0 = \sqrt{-g/2}e^{g|x|/2}$ and this ground state wavefunction is the only one of the bound spectrum at the energy $E_0 = -g^2/4$. Thus, this state will be deleted from the spectrum of the partner potential, which is purely repulsive. However, the situation is by far the more interesting in the case of the strictly isospectral construction as one can see in the following.

A simple calculation shows that

$$\mathcal{I}(x) = \int_{-\infty}^{x}\psi_0^2(x')dx' = -\frac{1}{2}\mathrm{sign}(x)e^{g|x|} + \frac{\mathrm{sign}(x)}{2} + \frac{1}{2}\ . \tag{5}$$

Thus one gets

$$V_{iso} = g\delta_{iso}(x) = g\delta(x) + 2g^2\frac{\mathcal{C}\mathrm{sign}(x)e^{-g|x|}}{(1-\mathcal{C}\mathrm{sign(x)}e^{-g|x|})^2} \tag{6}$$

and the isospectral wavefunction reads

$$\psi_{0,iso} = -\sqrt{-2g}\sqrt{C(C+1)}\frac{\mathrm{sign}(x)e^{-g|x|/2}}{(1-\mathcal{C}\mathrm{sign(x)}e^{-g|x|})}\ , \tag{7}$$

where $\mathcal{C} = 2C + \mathrm{sign}(x) + 1$. The eigenvalue corresponding to the isospectral wavefunction is the same as for the common delta bound state, i.e., $E_0 = -g^2/4$. The analysis of Eqs. (6) and (7) shows that possible singularities are to be found for C in the interval (-1,-1/2], which is excluded when one considers normalizable isospectral wavefunctions. However for non-normalizable solutions these singularities should be taken into account.

The plots we did for the isospectral potentials as a function of the isospectral parameter (figure 1) display a shallow potential well on the negative half-line moving toward the origin where it is absorbed by the delta singularity there, and on the positive half-line a tail dying off at increasing C We also present plots showing the behaviour of the normalized isospectral wavefunctions for the same values of the C parameter as for the potentials (see figure 2). Moreover, figures 3 and 4 display the moving singularity structure when we do not introduce the normalization constant in equation (7). In summary, we believe that the strictly isospectral extension of the attractive δ potential introduced here may be relevant for many applications, once one allows for a physical origin of the C-dependence. For example, the parameter C may express the effect of static and/or moving distant boundaries, as well as sample-size dependence [10, 11]. If one does not discard as unphysical the non-normalizable isospectral solutions, one may think of the isospectral method as allowing to introduce singularities in both wavefunctions and potentials which apparently are required to explain the extra losses of ultracold neutrons at the walls [12].

Acknowledgments

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Figure Captions

Figure 1.

Darboux potential contributions for C equal to .00001, .10001, 1.10001, 5.10001, for $g = -1$ from left to right.

Figure 2.

The corresponding isospectral wavefunctions for the same values of C as in figure 1 showing how in the infinite limit of C one recovers the original δ wavefunction. Actually, for rather low values of C, the isospectral δ wavefunction is already very close in shape to the original one.

Figure 3.

Darboux potential contributions for C equal to -1.4, -0.9, (up), and, -0.6, -0.3 (down) for $g = -1$.

Figure 4.

Non-normalizable isospectral wavefunctions for the same values of C as in figure 3, together with the original ground state δ wavefunction displayed in the first plot of the figure ($g = -1$).

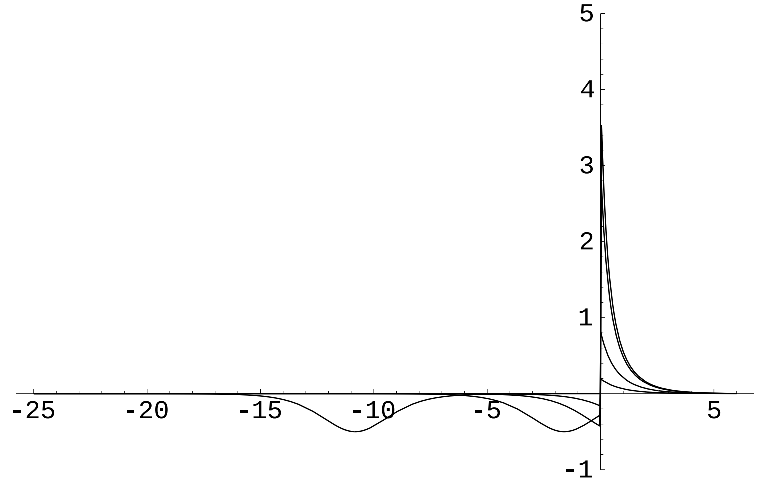

Figure 1

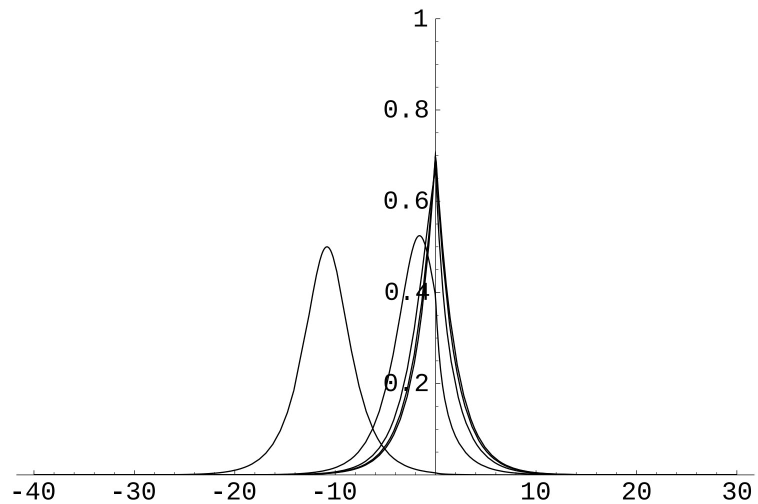

Figure 2

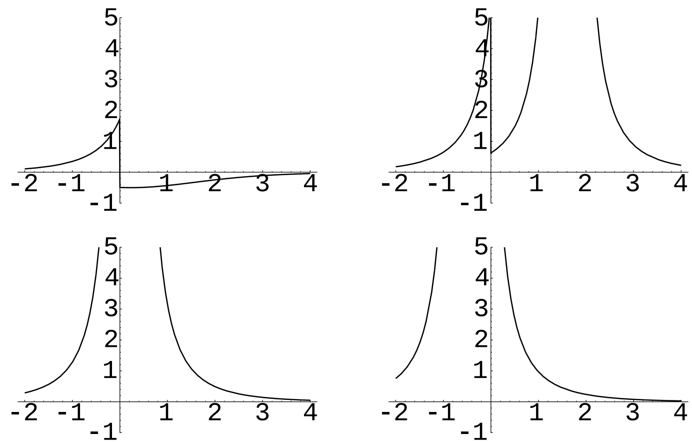

Figure 3

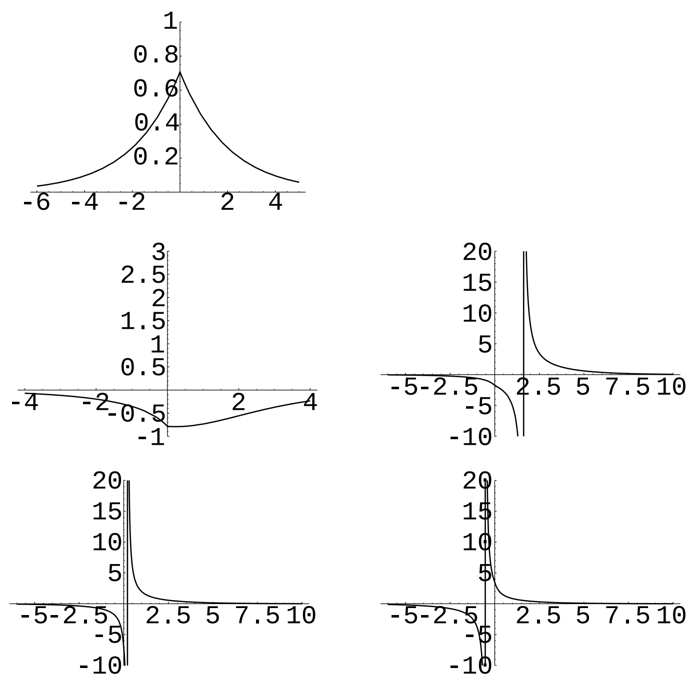

Figure 4